Analytic Study and Numerical Method of The Skew Lognormal Cascade Distribution

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Abstract

This working paper studies the "skew lognormal cascade distribution", which is proposed by the first time here, as the static solution of the simplified SIBM model (Stephen Lihn 2008, SSRN: 1149142). This distribution exhibits fat-tail, asymmetry tunable by a skew parameter, converges to the normal distribution, and has finite moments. These fine properties make it very useful in financial applications. The analytic formula of the raw moments and the cumulants are calculated for both the symmetric and skew forms. The implication to the multiscaling property is also studied for the symmetric distribution. The Taylor expansion on the distributions and their logarithms are carried out. A numeric method is carried out for the numerical computation of the probability density function. This method can be implemented via a computer algebra system and enable the numerical algorithm to produce high precision result. This distribution is implemented on http://www.skew-lognormal-cascade-distribution.org/ by the author. The author has tried to apply the distribution to the daily log returns of several financial time series, such as DJIA, WTI spot oil, XAU index, VIX index, 10-year Treasury, and several currencies. They all showed very good fit.

Introduction. This short working paper studies the skew lognormal cascade distribution, which is the static solution of the simplified SIBM model (Lihn 2008, SSRN: 1149142). The skew distribution, which exhibits fat tail while maintains finite moments, can be a very useful distribution describing financial statistics, such as the stock market. The attempt is to work out the power expansion of this distribution so that the probability distribution function (pdf) can be computed numerically without using computationally expensive integrals. Even if this is not possible, the study should shed light on the inner structure of this distribution. Both the symmetric distribution and the skew distribution are studied. The analytic formula for the moments and cumulants are presented. The implication to the multiscaling property is studied for the symmetric distribution.

Complicated algebraic results in this paper are carried out by the open source computer algebra system, Maxima. Numerical computation is prototyped on GNU Octave.

1 The Simplified SIBM Model

In this section, we define the simplified SIBM model (Lihn 2008, SSRN: 1149142). We shall use a logarithmic model for continuous-time stock price processes. The stock price process X shall always be presented in its logarithm, $\log X$ (See 1.1 of Fernholz 2002), which is abbreviated as χ.

Under the limit of $\tau_c \to 0$, the SIBM model can be reduced to the following stochastic equation for for the stock price process:

$$d_t\chi(t) = \Phi \cdot e^{\mathcal{H}} \left[d_t W(t) + (\beta \cdot \mathcal{H} + g) \, dt \right]. \tag{1}$$

This expression could be very useful in finance since most stochastic equations in finance are written in the stock price processes, instead of the return processes. The parameters are defined as following: Φ is a global constant, β is "the skewness parameter", and g is the constant growth term. $\mathcal{H}$ is a time dependent normal process. However, to investigate the static return distribution of a fixed time lag T, we can assume $\mathcal{H}$ follows the higher order randomness hypothesis (HORN):

$$\mathcal{H} \sim N\left(0, \eta^2\right). \tag{2}$$

In the computational order, averaging on $\mathcal{H}$ is applied last. With the help of the HORN, we can define the log-price change $x = \chi(T) - \chi(0)$ and deduce the probability distribution function (pdf) $p(x)$ by simply observing the functional form of the normal distribution:

$$p(x) = \int_{-\infty}^{\infty} d\mathcal{H} \, \frac{1}{2\pi \, \eta \, \sigma(\mathcal{H})} e^{-\frac{\mathcal{H}^2}{2\eta^2}} e^{-\frac{(x - \sigma(\mathcal{H}) \, (\beta \cdot \mathcal{H} + g))^2}{2\sigma(\mathcal{H})^2}}, \tag{3}$$

where $\sigma(\mathcal{H}) = \Phi \cdot e^{\mathcal{H}}$. It is easy to verify that $\int_{-\infty}^{\infty} p(x)\, dx = 1$. This pdf is the same as the following distribution construct:

$$\mathcal{D}_{\eta,\beta,g,\Phi} = \{x = (a + \beta \cdot b + g) \cdot \Phi e^b, a \in N(0,1), b \in N(0,\eta^2)\}. \tag{4}$$

With the substitution of $h = \mathcal{H}/\eta$, Equation (3) becomes

$$p_1(x) = \int_{-\infty}^{\infty} dh\, \frac{1}{2\pi\, \sigma(h)} e^{-\frac{h^2}{2}} e^{-\frac{(x-\sigma(h)\ (\beta\eta h+g))^2}{2\sigma(h)^2}}, \tag{5}$$

where $\sigma(h) = \Phi \cdot e^{\eta h}$. When $\beta = 0$ and $g = 0$, we have the symmetric pdf of

$$p_2(x) = \int_{-\infty}^{\infty} dh\, \frac{1}{2\pi\, \sigma(h)} e^{-\frac{h^2}{2}} e^{-\frac{x^2}{2\sigma(h)^2}}. \tag{6}$$

In this paper, we will focus on exploring the properties of $p_1(x)$ and $p_2(x)$. The complexity of $p_2(x)$ is less than $p_1(x)$, thus we shall work on $p_2(x)$ first.

2 The Properties of the Symmetric Distribution

In this section, we attempt to work out the statistic properties for the symmetric case. First, by the substitution of $z = h + \eta$, we arrive at

$$p_2(x) = \int_{-\infty}^{\infty} dz\, \frac{1}{2\pi\Phi} e^{\frac{\eta^2}{2}-\frac{z^2}{2}} \exp(-\frac{x^2}{2\Phi^2} e^{-2(\eta z-\eta^2)}). \tag{7}$$

The characteristic function of $p_2(x)$ is $\Psi_2(t) = \int_{-\infty}^{\infty} dx\, e^{itx} p_2(x)$ and

$$\Psi_2(t) = \int_{-\infty}^{\infty} dz\, \frac{1}{\sqrt{2\pi}} e^{\frac{\eta^2}{2}-\frac{z^2}{2}} e^{\eta z-\eta^2} \exp(-\frac{\Phi^2 t^2}{2} e^{2(\eta z-\eta^2)}). \tag{8}$$

Applying Equation (58), we have

$$\Psi_2(t) = \sum_{k=0}^{\infty} H_{2k}(0) \frac{\Phi^{2k} t^{2k}}{2^k (2k)!} \int_{-\infty}^{\infty} dz\, \frac{1}{\sqrt{2\pi}} e^{\frac{\eta^2}{2}-\frac{z^2}{2}} e^{(2k+1)(\eta z-\eta^2)}. \tag{9}$$

For the symmetric distribution, all the odd terms are zero. The raw moments μ_n are defined in terms of the characteristic function: $\Psi_2(t) = \sum_{n=0}^{\infty} \mu_n (it)^n / n!$. The integral is simply $e^{2k^2\eta^2}$. Thus we have

$$\begin{aligned} \mu_{2k} &= \tfrac{|H_{2k}(0)|}{2^k} \Phi^{2k} e^{2k^2\eta^2} = \tfrac{(2k)!}{2^k k!} \Phi^{2k} e^{2k^2\eta^2}, \text{ or} \\ \mu_n &= |n-1|!!\Phi^n e^{\frac{n^2\eta^2}{2}} \quad (n \geq 0). \end{aligned} \tag{10}$$

Multiscaling: The functional form of Equation (10) is similar to that of a normal distribution $N(\mu, \sigma^2)$ if we substitute σ with $\Phi\, e^{k\eta^2}$. We can compare Equation (10) to the multiscaling of the return moments: $\mu_n = C_n T^{\zeta(n)}$ in the multifractal model. We get that $C_{2k} = \frac{(2k)!}{2^k k!}$ is simply the prefactor of the nth moment at scale $T = 1$ and $\zeta(n)$ is the exponent of multiscaling power law. It indicates that, in order for multiscaling to hold, we must have

$$\zeta(n)\ln(T) = \frac{n^2}{2}\,\eta^2(T) + n\ln(\Phi(T)). \tag{11}$$

Thus $\eta^2(T) = -2a\ln(T)$, $\Phi(T) = T^b$ where a, b are constants, and $\zeta(n) = -an^2 + b\,n$. The constant b resembles the Hurst exponent in the fractional Brownian motion. This is well understood. But the case for η^2 is more complicated. We know that, in the financial markets, η^2 is positive and approaches to zero as T increases. Therefore, we must have $\ln(T) < 0$ and $a > 0$. η^2 scales linearly with $\ln(T)$. This is similar to how the autocorrelation of the volatility scales. We also notice that η^2 causes $\zeta(n)$ to have a second order scaling term n^2.

The cumulants κ_n are related to the raw moments μ_n by

$$\sum_{n=0}^{\infty}\mu_n\frac{(it)^n}{n!} = \exp\left(\sum_{n=0}^{\infty}\kappa_n\frac{(it)^n}{n!}\right). \tag{12}$$

This formula allows us to derive all the cumulants from the raw moments. Similar to the Hermite Polynomials, we can define the polynomials $L_{2k}(x)$ for the cumulants in the symmetric lognormal cascade distribution such that

$$\kappa_{2k} = \frac{(2k)!}{2^k k!}\Phi^{2k}e^{2k\eta^2}L_{2k}(e^{4\eta^2}). \tag{13}$$

Table 1 shows the first few $L_{2k}(x)$ polynomials. There are two parts in $L_{2k}(x)$: The first part is $(x-1)^{k-1}$. The second part is a polynomial in the order of $(k-1)(k-2)/2$ when $k > 2$. The constant in the second polynomial (i.e., $|L_{2k}(0)|$) is simply $(k-1)!$.

Table 1: The polynomials $L_{2k}(x)$ for the cumulants in the symmetric lognormal cascade distribution

$L_0(x) = 0$
$L_2(x) = 1$
$L_4(x) = x - 1$
$L_6(x) = (x-1)^2(x+2)$
$L_8(x) = (x-1)^3(x^3 + 3x^2 + 6x + 6)$
$L_{10}(x) = (x-1)^4(x^6 + 4x^5 + 10x^4 + 20x^3 + 30x^2 + 36x + 24)$
$L_{12}(x) = (x-1)^5(x^{10} + 5x^9 + 15x^8 + 35x^7 + 70x^6 + 120x^5 + 180x^4 + 240x^3 + 270x^2 + 240x + 120)$

The second cumulant is the variance, $\kappa_2 = \mu_2 = \Phi^2 e^{2\eta^2}$; and the fourth cumulant equals to ($\mu_4 = 3\Phi^4 e^{8\eta^2}$)

$$\kappa_4 = 3\Phi^4 e^{4\eta^2}(e^{4\eta^2} - 1). \tag{14}$$

The kurtosis is $\kappa_4/\mu_2^2 = 3(e^{4\eta^2} - 1)$. The kurtosis increases rapidly as η increases. For example, the stock market often has $\eta \approx 0.5$. The change in η can change the kurtosis by nearly one order of magnitude. The kurtosis is 2.6 when $\eta = 0.4$, but it is 5.1 if $\eta = 0.5$, and it reaches 9.7 when $\eta = 0.6$. So it is quite difficult to decide the exact value of η from the observed kurtosis in the data.

Multiscaling: On the other hand, if we consider multiscaling law in Equation (11), we have the kurtosis equal to $3(T^{-8a} - 1)$ where $a > 0$ and $T < 1$. In the meantime, the variance scales like T^{-4a+2b}. We find that, as T increases, kurtosis decreases much faster than the variance.

We shall also calculate the sixth cumulant. From $\mu_6 = 15\Phi^6 e^{18\eta^2}$ and

$$\kappa_6 = 15\Phi^6 e^{6\eta^2}(e^{12\eta^2} - 3e^{4\eta^2} + 2). \tag{15}$$

And for the eighth cumulant, from $\mu_8 = 105\Phi^8 e^{32\eta^2}$, we have

$$\kappa_8 = 105\Phi^8 e^{8\eta^2}(e^{24\eta^2} - 4e^{12\eta^2} - 3e^{8\eta^2} + 12e^{4\eta^2} - 6). \tag{16}$$

This result will be used in Section 3 to verify the relation between the cumulants and the Taylor expansion of $\ln(p_2(x))$.

3 The Taylor Expansion of the Symmetric Distribution

It is immediately obvious that $p_2(0) = \frac{1}{\sqrt{2\pi}\Phi} e^{\eta^2/2}$. And $dp_2(0)/dx = 0$. Thus $p_2(x)$ has a round top near zero. In general, we use Taylor expansion to obtain $d^n p_2(0)/dx^n$ which is denoted as $p_2^{(n)}(0)$. After we work out $p_2^{(n)}(0)$, we will proceed to obtain Taylor series of $\ln(p_2(x))$ which is more useful in numerical applications.

Starting from Equation (7), we have the Taylor series of $p_2(x)$ as

$$p_2(x) = \int_{-\infty}^{\infty} dz \, \frac{1}{2\pi\Phi} e^{\frac{\eta^2}{2} - \frac{z^2}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k! \Phi^{2k}} e^{-2k(\eta z - \eta^2)}, \tag{17}$$

which is

$$p_2(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k! \Phi^{2k+1}} e^{\frac{\eta^2}{2}(2k+1)^2}. \tag{18}$$

This should equate to the Taylor expansion of $p_2(x) = \sum_{n=0}^{\infty} p_2^{(n)}(0) \frac{x^n}{n!}$. Therefore, we get

$$p_2^{(2k)}(0) = \frac{1}{\sqrt{2\pi}} \frac{(-1)^k (2k)!}{2^k k! \Phi^{2k+1}} e^{\frac{\eta^2}{2}(2k+1)^2}, \tag{19}$$

which is also equal to $\frac{1}{\sqrt{2\pi}}\frac{(-1)^k}{\Phi^{2k+1}}e^{\frac{\eta^2}{2}}\mu_2^k\mu_{2k}$.

The more interesting presentation with regards to the lognormal cascade distribution is $\ln(p_2(x))$, which is reduced to the form of a normal distribution, $-x^2$ when $\eta \to 0$. We can define

$$\begin{aligned} \ln\left(\frac{p_2(x)}{p_2(0)}\right) &= \sum_{n=1}^{\infty} \frac{\mathcal{L}_n x^n}{n!}, \\ \mathcal{L}_n &= \frac{d^{n-1}}{dx^{n-1}}\left(\frac{p_2^{(1)}(x)}{p_2(x)}\right)_{x=0}. \end{aligned} \tag{20}$$

All $\mathcal{L}_n$ terms can be calculated since all $p_2^{(n)}(0)$ terms are known via Equation (19). It turns out that the Taylor expansion of $\ln(p_2(x))$ is directly related to the polynomials in the cumulants as

$$\mathcal{L}_{2k} = (-1)^k(2k-1)!!\Phi^{-2k}e^{4k\eta^2}L_{2k}(e^{4\eta^2}), \tag{21}$$

or $(-1)^k\kappa_2^k\kappa_{2k}\Phi^{-6k}$. This relation gives us an alternative way of computing $L_{2k}(x)$ to as high order as we wish. For instance, $\mathcal{L}_2 = -\Phi^{-2}e^{4\eta^2} = -\kappa_2^2\Phi^{-6}$; $\mathcal{L}_4 = 3\Phi^{-4}e^{8\eta^2}(e^{4\eta^2}-1) = \kappa_2^2\kappa_4\Phi^{-12}$; $\mathcal{L}_6 = -15\Phi^{-6}e^{12\eta^2}(e^{12\eta^2} - 3e^{4\eta^2} + 2)$, which is $-\kappa_2^3\kappa_6\Phi^{-18}$.

Limitation of Taylor Expansion Method: Since $\ln(\frac{p_2(x)}{p_2(0)})$ should be negative all the time, the Taylor series is better to stop at a negative term to prevent from blowing up. Thus $k_{\max}$ should be an odd number. However, numerical simulation has shown that the methods of Taylor expansion outlined in this section do not go very far. In particular, it does not work well for large η and large x. The problem of divergence becomes obvious when we recognize that L_{2k} is in the order of $(k-1)(k-2)/2$ in $e^{4\eta^2}$. Thus the power series diverges badly as k increases. We are in need of another method for numerical computation.

4 The Numerical Method for The Symmetric Distribution

Equation (20) on the surface looks like a fairly good analytic solution for the symmetric distribution. But when we use it for numerical computation, we find that Taylor series does not converge fast enough as η increases from zero. In particular, when $\eta \to 0.5$, it would require many orders to compute $\ln(p_2(x))$. Thus the Taylor expansion on $\ln(p_2(x))$ becomes useless. That is, we have a singularity point in this distribution which deserves some study. Especially $\eta = 0.5$ represents the characteristic fits of the US stock market and the tails appear to be linear, which resembles a power law. It seems to have a special meaning.

The divergence stems from the fact that the e^{η^2} term is in the order of k^2 in this distribution. This is very unusual in Taylor expansion.

Since the logarithmic expansion is invalid, let's go back to Equations (8) and investigate $p_2(x)$ in more detail. We can decompose the integrand in terms of the exponentials of two polynomials, $f_1(z)$ and $f_2(z)$.

$$p_2(x) = \int_{-\infty}^{\infty} dz\, \frac{1}{2\pi\Phi} e^{\frac{\eta^2}{2}} e^{f_1(z)} e^{f_2(z)}, \tag{22}$$

in which

$$f_1(z) = -\frac{z^2}{2}, f_2(z) = -\frac{x^2}{2\Phi^2} e^{-2\eta z + 2\eta^2}. \tag{23}$$

$f_1(z)$ does not vary with x and η. It diverges to $-\infty$ quadratically and is symmetric on z. Only $f_2(z)$ depends on x and η. It converges to zero when z is large and diverges to $-\infty$ exponentially as z decreases. Therefore, the sum of $f_1(z)$ and $f_2(z)$ defines a region, smaller than that of $f_1(z)$, to be integrated to yield $p_2(x)$. The peak of $f_1(z) + f_2(z)$ can be determined by $\left(\frac{d(f_1(z)+f_2(z))}{dz}\right)_{z=\,z_p} = 0$ and

$$z_p = \frac{\eta x^2}{\Phi^2} e^{2\eta^2} e^{-2\eta\, z_p}. \tag{24}$$

We can define $c = 2\eta \left(\frac{\eta x^2}{\Phi^2} e^{2\eta^2}\right)$ and $z_p = \frac{c\,y}{2\eta} = y \left(\frac{\eta x^2}{\Phi^2} e^{2\eta^2}\right)$, and reduce Equation (24) to the following mathematical problem

$$y = e^{-c\,y} \quad \text{or} \quad \frac{1}{y} \ln(\frac{1}{y}) = c,\ (c > 0). \tag{25}$$

This can be solved numerically and the resulting $z_p(x, \eta)$ determines the order of magnitude of $p_2(x)$. The next task is to determine the width of $e^{f_1(z)} e^{f_2(z)}$ around z_p, which can be accomplished by Taylor expansion at z_p. Let $z = z_p + v$, we have

$$f_2(z_p + v) = -\frac{z_p}{2\eta} e^{-2\eta v} \approx -\frac{z_p}{2\eta}\left(1 - 2\eta v + 2\eta^2 v^2\right) + f_3(v) \tag{26}$$

where $f_3(v) = -\frac{z_p}{2\eta} \sum_{k=3}^{\infty} \frac{(-2\eta)^k v^k}{k!}$ contains the higher order terms. This expansion allows us to obtain

$$p_2(x) = \int_{-\infty}^{\infty} dz\, \frac{1}{2\pi\Phi} e^{\frac{\eta^2}{2}} e^{-\frac{1}{2}(z_p+v)^2} e^{-\frac{z_p}{2\eta}\left(1-2\eta v+2\eta^2 v^2\right)} e^{f_3(v)}, \tag{27}$$

which is

$$p_2(x) = \frac{1}{2\pi\Phi} e^{\frac{1}{2}\left(\eta^2 - \frac{z_p}{\eta} - z_p^2\right)} \int_{-\infty}^{\infty} dz\, e^{-\frac{1}{2}\psi^2 v^2} e^{f_3(v)}, \tag{28}$$

where $\psi = \sqrt{1 + 2\eta z_p}$. From Equation (28), we found that knowing z_p is very useful in the calculation of $p_2(x)$. z_p is where the maximum probability is located. And $\sigma = 1/\psi$ is approximately the "width" of the Gaussian integrand. If $p_2(x)$ is to be carried out numerically, one only needs to integrate in the neighborhood of z_p extending reasonably far from the "width". More details on numerical integration will be explored when we discuss the numerical method of the skew distribution.

Ignoring $f_3(v)$, we have the first order solution

$$p_2(x) = \frac{1}{\sqrt{2\pi\Phi}} e^{\frac{1}{2}\left(\eta^2 - \frac{z_p}{\eta} - z_p^2\right)} \frac{1}{\psi}. \quad (29)$$

When both η and $\eta^2 x^2$ are very small, $z_p \approx \frac{\eta x^2}{\Phi^2}$ and $p_2(x) \approx \frac{1}{\sqrt{2\pi\Phi}} e^{-\frac{1}{2}\left(\frac{x^2}{\Phi^2}\right)}$, which demonstrates the convergence to a normal distribution.

To calculate the higher order terms, we can expand $e^{f_3(v)} = \sum_{k=0}^{\infty} a_k v^k$ where $a_k = \frac{1}{k!}\frac{d^k}{dx^k}\left(e^{f_3(v)}\right)_{v=0}$, and use Equation (64) to carry out the Gaussian integrals. Notice that $a_0 = 1$ and $a_1 = a_2 = 0$. The Gaussian integrals on odd v^k terms are zero. Thus we have

$$p_2(x) = \frac{1}{\sqrt{2\pi\Phi}} e^{\frac{1}{2}\left(\eta^2 - \frac{z_p}{\eta} - z_p^2\right)} \left(\frac{1}{\psi} + \sum_{k=2}^{\infty} a_{2k} \frac{|2k-1|!!}{\psi^{2k+1}}\right). \quad (30)$$

The first few a_{2k} are listed in Table 2 . This formula does satisfy our need for a more accurate numerical calculation over a large range of η and x. For instance, when $\eta = 0.5$, Equation (30) has precision of more than 95% when expanded up to a_{10}. However, it still does not provide any insight about the tail structure of $p_2(x)$.

Table 2: The parameters a_k in the Taylor series of $e^{f_3(v)}$ in the symmetric lognormal cascade distribution (And $3!! = 3$, $5!! = 15$, $7!! = 105$, $9!! = 945$)

$a_4 = -z_p\eta^3/3$
$a_6 = z_p\eta^4(-2\eta + 10z_p)/45$
$a_8 = z_p\eta^6(-2\eta + 91z_p)/630$
$a_{10} = z_p\eta^8(-2\eta + 456z_p - 1050\frac{z_p^2}{\eta})/14175$

5 The Properties of the Skew Distribution

In this section, we follow the same approach to solve the more complicated skew distribution

$$p_1(x) = \int_{-\infty}^{\infty} dh\, \frac{1}{2\pi\,\sigma(h)} e^{-\frac{h^2}{2}} e^{-\frac{(x-f(h))^2}{2\sigma(h)^2}}, \quad (31)$$

where $f(h) = \sigma(h)\,(\beta\eta h + g)$ and $\sigma(h) = \Phi e^{\eta h}$. By the substitution of $z = h + \eta$, we arrive at

$$p_1(x) = \int_{-\infty}^{\infty} dz\, \frac{1}{2\pi\Phi} e^{\frac{\eta^2}{2}-\frac{z^2}{2}} \exp(-\frac{(x-f(z-\eta))^2}{2\Phi^2} e^{-2(\eta z-\eta^2)}). \tag{32}$$

The characteristic function of $p_1(x)$ is $\Psi_1(t) = \int_{-\infty}^{\infty} dx\, e^{itx} p_1(x)$ and

$$\Psi_1(t) = \int_{-\infty}^{\infty} dz\, \frac{1}{\sqrt{2\pi}} e^{\frac{\eta^2}{2}-\frac{z^2}{2}} e^{\eta z-\eta^2} \exp(it\, f(z-\eta) - \frac{\Phi^2 t^2}{2} e^{2(\eta z-\eta^2)}). \tag{33}$$

Let $m_z = f(z-\eta)/\Phi\sigma_z = (\beta\eta z - \beta\eta^2 + g)$ and $\sigma_z = e^{\eta z-\eta^2}$, using Hermite polynomials, we have

$$\Psi_1(t) = \int_{-\infty}^{\infty} dz\, \frac{1}{\sqrt{2\pi}} e^{\frac{\eta^2}{2}-\frac{z^2}{2}} \sigma_z \sum_{n=0}^{\infty} H_n(im_z/\sqrt{2}) \frac{(\Phi t\sigma_z/\sqrt{2})^n}{n!}. \tag{34}$$

The nth moment of $p_1(x)$, μ_n, is the coefficients of $(it)^n$ multiplied by $(n!)$ from the Taylor series of $\Psi_1(t)$. That is, $d^n\Psi_1(t)/i^n dt^n$, which is

$$\mu_n = \int_{-\infty}^{\infty} dz\, \frac{\Phi^n}{\sqrt{2\pi}} e^{\frac{\eta^2}{2}-\frac{z^2}{2}} \sigma_z^{n+1} \frac{H_n(im_z/\sqrt{2})}{(i\sqrt{2})^n}, \tag{35}$$

which can be simplified by letting $z = w + (n+1)\eta$,

$$\mu_n = \Phi^n e^{\frac{n^2\eta^2}{2}} \int_{-\infty}^{\infty} dw \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \sum_{k=0}^{n} |H_n^k| \frac{(\beta\eta)^k (w+\beta_n/\beta\eta)^k}{(\sqrt{2})^{n+k}}, \tag{36}$$

in which $\beta_n = n\beta\eta^2 + g$. By applying Equation (66) $(a = 1/2, b = \beta_n/\beta\eta)$, we obtain the analytical form of the raw moments

$$\mu_n = \Phi^n e^{\frac{n^2\eta^2}{2}} \sum_{k=0}^{n} \frac{|H_n^k|}{(\sqrt{2})^{n+k}} \sum_{p=0}^{[k/2]} \frac{k!|2p-1|!!}{(k-2p)!(2p)!} \beta_n^{k-2p} (\gamma-1)^p, \tag{37}$$

where $[f]$ is the largest integer not greater than f and $\gamma = \beta^2\eta^2 + 1$.

The first raw moment is straightforward

$$\mu_1 = \Phi e^{\frac{\eta^2}{2}} \beta_1 = \Phi e^{\frac{\eta^2}{2}} (\beta\eta^2 + g). \tag{38}$$

The second raw moment is

$$\mu_2 = \Phi^2 e^{2\eta^2} (\gamma + \beta_2^2). \tag{39}$$

The third raw moment is

$$\mu_3 = \Phi^3 e^{\frac{9\eta^2}{2}} \beta_3 (3\gamma + \beta_3^2). \tag{40}$$

The fourth raw moment is

$$\mu_4 = \Phi^4 e^{8\eta^2}(3\gamma^2 + 6\beta_4^2\gamma + \beta_4^4). \tag{41}$$

The cumulants can be calculated via Equation (12). The first cumulant κ_1 is the same as the first raw moment μ_1, which is the mean of the skew distribution. We can clearly see that both g and $\beta\eta^2$ shift the distribution, with the prefactor of $2\Phi e^{\frac{\eta^2}{2}}$. If we were to require a positive mean while having a negative skewness, which is the case in the stock market, we then need $g > -\beta\eta^2$ ($\beta < 0$). However, the growth term g is part of β_n in the higher moments. The factor n in β_n makes g less important as n increases. The second cumulant κ_2 is $\mu_2 - \mu_1^2$, which is the variance. The third cumulant κ_3 is $\mu_3 - 3\kappa_2\kappa_1 - \kappa_1^3$. The skew is defined as $\kappa_3/\kappa_2^{3/2}$. The fourth cumulant κ_4 is $\mu_4 - 4\kappa_3\kappa_1 - 3\kappa_2^2 - 6\kappa_2\kappa_1^2 - 1\kappa_1^4$. And the kurtosis is κ_4/κ_2^2. The analytical form of cumulants has become very complicated, thus is not listed here.

6 The Taylor Expansion of the Skew Distribution

In the skew distribution $p_1(x)$, all the odd moments and odd derivatives are nonzero. For example, as a baseline, $p_1(0)$ is

$$p_1(0) = \frac{1}{\sqrt{2\pi}\Phi}\frac{1}{\gamma^{1/2}}e^{\frac{-(2\beta g+1)\eta^2+g^2}{2\gamma}}. \tag{42}$$

And the first derivative is

$$p_1^{(1)}(0) = \frac{-1}{\sqrt{2\pi}\Phi^2}\frac{3\beta\eta^2 - \beta_1}{\gamma^{3/2}}e^{-\frac{5\beta^2\eta^4 - 2(3\beta\beta_1+2)\eta^2+\beta_1^2}{2\gamma}}. \tag{43}$$

We get a taste of complexity involved in the skew distribution. In general, we use Taylor expansion to obtain $d^n p_1(0)/dx^n$ which is denoted as $p_1^{(n)}(0)$. After we work out $p_1^{(n)}(0)$, we then proceed to obtain Taylor series of $\ln(p_1(x))$.

Starting from Equation (32) and $m_z = (\beta\eta z - \beta\eta^2 + g)$ and $\sigma_z = e^{\eta z - \eta^2}$, we have

$$p_1(x) = \int_{-\infty}^{\infty} dz\, \frac{1}{2\pi\Phi}e^{\frac{\eta^2 - z^2}{2}}\exp(-\frac{1}{2}(\frac{x}{\sigma_z\Phi} - m_z)^2), \tag{44}$$

which is expanded into

$$p_1(x) = \int_{-\infty}^{\infty} dz\, \frac{1}{2\pi\Phi}e^{\frac{\eta^2 - z^2 - m_z^2}{2}}\sum_{n=0}^{\infty}\frac{\mathbb{M}_n(m_z, i)}{n!}\left(\frac{x}{\sigma_z\Phi}\right)^n. \tag{45}$$

This should equate to the Taylor expansion of $\sum_{n=0}^{\infty} p_1^{(n)}(0)\frac{x^n}{n!}$.

Therefore, we get

$$p_1^{(n)}(0) = \int_{-\infty}^{\infty} dz\, \frac{1}{2\pi\Phi^{n+1}} e^{\frac{\eta^2-z^2-m_z^2}{2}} \sigma_z^{-n} \mathbb{M}_n(m_z, i), \tag{46}$$

which can be simplified by letting $w = z + \eta c_{w,n}$,

$$p_1^{(n)}(0) = \frac{e^{a_{w,n}}}{2\pi\Phi^{n+1}} \int_{-\infty}^{\infty} dw\, e^{-\frac{w^2}{2}} \mathbb{M}_n(\, m_w, i), \tag{47}$$

in which $c_{w,n} = ((n+1) + \beta g - \gamma)/\gamma$, $a_{w,n} = ((n+1)^2\eta^2 + 2(n+1)g\beta\eta^2 - g^2)/(2\gamma^2)$, and $m_w = \beta\eta w - (c_{w,n}+1)\beta\eta^2 + g$. It is now obvious that $p_1^{(n)}(0)$ in Equation (47) can be carried out by a computer algebra system, similar to Equation (36).

Thus the Taylor expansion of the skew lognormal cascade distribution in its logarithm, $\ln(p_1(x))$ can also be calculated correspondingly.

$$\begin{aligned} \ln(p_1(x)) &= \ln(p_1(0)) + \sum_{n=1}^{\infty} \tfrac{\mathcal{L}_n x^n}{n!}, \\ \mathcal{L}_n &= \tfrac{d^{n-1}}{dx^{n-1}} \left(\tfrac{p_1^{(1)}(x)}{p_1(x)} \right)_{x=0}. \end{aligned} \tag{48}$$

We can calculate all $\mathcal{L}_n$ terms since all $p_1^{(n)}(0)$ terms are known via Equation (47). Here we only work out the first few terms: $\mathcal{L}_1 = \frac{g-2\beta\eta^2}{\gamma\Phi} e^{(3+2\beta g)\eta^2/2\gamma}$; $\mathcal{L}_2 = \frac{g-2\beta\eta^2}{(\gamma\Phi)^2} e^{(3+2\beta g)\eta^2/2\gamma} (((g-3\beta\eta^2)^2 - \gamma)e^{\eta^2/\gamma} - (g-2\beta\eta^2)^2)$; $\mathcal{L}_4 = 3\Phi^{-4} e^{8\eta^2}(e^{4\eta^2}-1) = \kappa_2^2\kappa_4\Phi^{-12}$; $\mathcal{L}_6 = -15\Phi^{-6}e^{12\eta^2}(e^{12\eta^2} - 3e^{4\eta^2} + 2)$, which is $-\kappa_2^3\kappa_6\Phi^{-18}$. That is, the Taylor expansion of $\ln(p_2(x))$ is related to the cumulants as $\mathcal{L}_{2k} = (-1)^k \kappa_2^k \kappa_{2k} \Phi^{-6k}$.

7 The Numerical Method of The Skew Distribution

In the similar manner as outlined in Section 4, we can obtain an efficient numerical method for the skew distribution. Let's go back to Equations (32) and investigate $p_1(x)$ in more detail. We can decompose the integrand in terms of the exponentials of two polynomials, $q_1(z)$ and $q_2(z)$.

$$p_1(x) = \int_{-\infty}^{\infty} dz\, \frac{1}{2\pi\Phi} e^{\frac{\eta^2}{2}} e^{q_1(z)} e^{q_2(z)}, \tag{49}$$

in which

$$q_1(z) = -\frac{z^2}{2}, q_2(z) = -\frac{(x - f(z-\eta))^2}{2\Phi^2} e^{-2\eta z + 2\eta^2}. \tag{50}$$

where $f(h) = \Phi e^{\eta h}(\beta\eta h + g)$. The addition of the skew term $f(z-\eta)$ only affects the peak of $q_1(z) + q_2(z)$ slightly (except for very large g). The formula describing the peak position z_p however becomes quite complicated due to the presence of $f(z-\eta)$

$$z_p = \frac{\eta}{\Phi^2} e^{2\eta^2} e^{-2\eta\, z_p} \Bigg((x - f(\, z_p - \eta))^2 + (x - f(\, z_p - \eta))\, \frac{f^{'}(\, z_p - \eta)}{\eta} \Bigg), \tag{51}$$

where $f^{'}(h) = \Phi e^{\eta h} \eta(\beta \eta h + \beta + g)$. The peak position z_p can be solved numerically and the resulting $z_p(x, \eta, \beta, g)$ determines the order of magnitude of $p_1(x)$. A precise z_p will significantly improve the accuracy of the integral. The next task is to determine the "width" of $e^{q_1(z)} e^{q_2(z)}$ around z_p, which can be accomplished by Taylor expansion at z_p. Let $z = z_p + v$, we have

$$q_1(z_p + v) + q_2(z_p + v) = -\frac{1}{2}\big(c_0 + \psi_1^2 v^2\big) + q_3(v) \tag{52}$$

where $c_0 = z_p^2 + \frac{z_p}{\eta} - \frac{1}{\Phi^2} e^{2\eta^2} e^{-2\eta\, z_p} (x - f(\, z_p - \eta))\, \frac{f^{'}(\, z_p - \eta)}{\eta}$, and ψ_1^2 is too complicated to be listed here. And $q_3(v) = \sum_{k=3}^{\infty} \frac{c_k v^k}{k!}$ contains the higher order terms ($c_k = \frac{d^k}{dv^k}(q_2(z_p + v))_{v=0}$). This expansion allows us to obtain

$$p_1(x) = \frac{1}{2\pi\Phi} e^{\frac{1}{2}\left(\eta^2 - c_0\right)} \int_{-\infty}^{\infty} dz\, e^{-\frac{1}{2}\psi_1^2 v^2} e^{q_3(v)}. \tag{53}$$

Ignoring $q_3(v)$, we have the first order solution

$$p_1(x) = \frac{1}{\sqrt{2\pi}\Phi} e^{\frac{1}{2}\left(\eta^2 - c_0\right)} \frac{1}{\psi_1}. \tag{54}$$

To calculate the higher order terms, we can expand $e^{q_3(v)} = \sum_{k=0}^{\infty} b_k v^k$ where $b_k = \frac{1}{k!} \frac{d^k}{dx^k}\left(e^{q_3(v)}\right)_{v=0}$, and use Equation (64) to carry out the Gaussian integrals. Notice that $b_0 = 1$ and $b_1 = b_2 = 0$. The Gaussian integrals on odd v^k terms are zero. Thus

$$p_1(x) = \frac{1}{\sqrt{2\pi}\Phi} e^{\frac{1}{2}\left(\eta^2 - c_0\right)} \left(\frac{1}{\psi_1} + \sum_{k=2}^{\infty} b_{2k} \frac{|2k-1|!!}{\psi_1^{2k+1}} \right). \tag{55}$$

Numerical integration can be carried out with high precision in the neighborhood of z_p by extending a reasonable range as determined by $\sigma \approx 1/\psi_1$. Numerically speaking, although ψ_1 is hard to calculate analytically, σ need not to be very precise. It only needs to be able to capture the "width" reasonably, which means $g(z_p \pm \sigma) \approx e^{-1/2}$. Then we can use, for instance, Simpson's rule to obtain $p_1(x)$. Assume we want to take m samples for each σ interval and perform our integration from $z_p - n\,\sigma$ to $z_p + n\,\sigma$, we have

$$p_1(x) \approx \left(4 \sum_{\text{even } k} g(z_k) + 2 \sum_{\text{odd } k} g(z_k) \right) \frac{h}{3}, \tag{56}$$

$$z_k = z_p + \frac{k}{m}\,\sigma,\ h = \frac{\sigma}{m},\ k\ \in [-m\,n, +m\,n],$$

in which $g(z)$ is the integrand

$$g(z) = \frac{1}{2\pi\Phi} e^{\frac{\eta^2}{2}} e^{q_1(z)+q_2(z)}. \tag{57}$$

When $m = 8$ and $n = 8$ and z_p is within 1% error to the real peak position, the integral requires 129 samplings (in addition to the effort of obtaining z_p and σ) and the result is within 10^{-8} error.

The numerical method has been implemented by the author using GNU Octave. It is made available on http://www.skew-lognormal-cascade-distribution.org/. The author has tried to apply the distribution to the daily log returns of several financial time series, such as DJIA, WTI spot oil, XAU index, VIX index, 10-year Treasury, and several currencies. They all showed very good fit.

8 Appendix: Hermite Polynomials, Gamma Function, Etc.

In this Appendix, we provide the definitions of Hermite polynomials and Gamma function according to Arfken 1985. These definitions are used extensively in this paper. We also suggest some approaches to solve $y = e^{-c\,y}$.

The **Hermite polynomials** $H_n(x)$ are defined by the generating function

$$\begin{aligned} g(x,t) = e^{-t^2+2tx} &= \sum_{n=0}^{\infty} H_n(x) \tfrac{t^n}{n!}, \\ H_n(x) &= \sum_{k=0}^{n} H_n^k x^k, \end{aligned} \tag{58}$$

where H_n^k are the coefficients in $H_n(x)$. Table 3 shows the first few such polynomials. $H_n(x)$ follow the recursive relations

$$\begin{aligned} H_{n+1}(x) &= 2x\, H_n(x) - 2n\, H_{n-1}(x), \\ H_n'(x) &= 2n\, H_{n-1}(x), \end{aligned} \tag{59}$$

which can be initiated by $H_0(x) = 1$ and $H_1(x) = 2x$. Useful special cases are $H_{2k}(0) = (-1)^k \frac{(2k)!}{k!}$ and $H_{2k+1}(0) = 0$. Alternatively, we may encounter the following form where all the coefficients are positive

$$\begin{aligned} g_2(x,t) = e^{t^2+2tx} &= \sum_{n=0}^{\infty} \tfrac{H_n(ix)}{i^n} \tfrac{t^n}{n!}, \\ \tfrac{H_n(ix)}{i^n} &= \sum_{k=0}^{n} \left| H_n^k \right| x^k. \end{aligned} \tag{60}$$

Table 3: First Seven Hermite Polynomials

$H_0(x) = 1$
$H_1(x) = 2x$
$H_2(x) = 4x^2 - 2$
$H_3(x) = 8x^3 - 12x$
$H_4(x) = 16x^4 - 48x^2 + 12$
$H_5(x) = 32x^5 - 160x^3 + 120x$
$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$

$H_n(x)$ is orthogonal to one another as following

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} H_n(x) H_m(x) = \delta_{mn}\, 2^n \sqrt{\pi}\, n!. \tag{61}$$

The **Gamma function** $\Gamma(z)$ is defined as a Gaussian integral

$$\Gamma(z) = 2\int_0^{\infty} dx\, e^{-x^2} x^{2z-1}. \tag{62}$$

When z is an integer, we have $\Gamma(z+1) = z!$. We occasionally use the double factorial notation:

$$\begin{gathered} (2n)!! = 2^n n! = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n), \\ (2n+1)!! = (2n+1)!/(2^n n!) = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1). \end{gathered} \tag{63}$$

The following **Gaussian integrals** are of importance

$$\begin{gathered} \int_0^{\infty} dx\, e^{-a\, x^2} x^{2p+1} = \frac{p!}{2a^{p+1}}, \\ \int_{-\infty}^{\infty} dx\, e^{-a\, x^2} x^{2p} = \frac{(p-1/2)!}{a^{p+1/2}} = \frac{|2p-1|!!}{(2a)^p}\sqrt{\frac{\pi}{a}}, p \geq 0. \end{gathered} \tag{64}$$

And note that $\Gamma(1/2) = \sqrt{\pi}$ and $\int_{-\infty}^{\infty} dx\, e^{-a\, x^2} = \sqrt{\frac{\pi}{a}}$. The binomial expansion is defined as

$$(x+b)^k = \sum_{p=0}^{k} C_k^p x^p b^{k-p}, \tag{65}$$

where C_k^p are the **binomial coefficients**, $\frac{k!}{(k-p)!p!}$. Thus

$$\begin{aligned} G(a,b,k) &= \int_{-\infty}^{\infty} dx\, e^{-a\, x^2} (x+b)^k \\ &= \sqrt{\frac{\pi}{a}} \sum_{p=0}^{[k/2]} \frac{k!|2p-1|!!}{(k-2p)!(2p)!} \frac{b^{k-2p}}{(2a)^p}, \end{aligned} \tag{66}$$

where $[f]$ is the largest integer not greater than f.

Numerical Solution for $y = e^{-c\,y}$:

The following equation

$$y(c) = e^{-c\,y(c)} \tag{67}$$

plays an important role to the solution of the lognormal cascade distribution. Thus we provide our exploration on it. In general, this equation can be solved via traditional root finding methods, such as Newton's method and the bisection root-finding method. These approaches are required to converge quickly in order to be useful in numerical implementation. Here we suggest three iterative approaches to find the root because they reveal important inner structure of this equation.

There are a few important known solutions of $y(c)$: when $c = 0$, $y = 1$; when $c = \ln(4)$ (about 1.386), $y = 1/2$; when $c = e$ (about 2.718), $y = 1/e$; when $c = 1$, $y = 0.567$ (that is, solution of $(1/y)^{1/y} = e$); when $c \to \infty$, $y = 0$. However, there is a singularity in this equation, that is, when $c = e$, $y = 1/e$. This singularity stems from the fact that if one attempts to use Newton's method, the first derivative becomes zero, that is, $ce^{-c\,y} - 1 \to 0$. Therefore, the convergence becomes infinitely slow near $c = e$.

When $c \ll e$ (for instance, $c < 0.1$), we can define $y_1 = c$ and apply $y_k = e^{-c\,y_{k-1}}$ iteratively. y_k will converge to the root quickly.

When c is reasonably large (for instance, $c > 0.1$), we can let $c = e^{1+p}$ and $y = e^{-(1+q)}$ and we have $1 + q = e^p e^{-q}$. Expanding the right hand side to the order of q^2, we can get the approximate root q (to less than 1% error) by solving $1 + q = e^p(1 - q + q^2/2)$. This approximation works well for small p ($p < 1$). For larger p, the initial guess can simply be $q_1 = p - \ln(1 + p)$, based on $q = p - \ln(1 + q)$. Then apply Newton's method $q_k = (e^{p-q_{k-1}}(1 + q_{k-1}) - 1)/(e^{p-q_{k-1}} + 1)$ to improve the accuracy of the root quickly. This method works quite well for a large range of c.

If $c \gg e$ (for instance, $c > e^{20}$), another simple alternative becomes available. That is to iterate on the inverse of y. Let $z = 1/y$, we have $z = c/\ln(z)$. We can define $z_1 = e^{1+q_1}$ and apply $z_k = c/\ln(z_{k-1})$ iteratively. z_k will converge to the inverse of the root quickly.

Once we use these methods to find out $y(c)$, we can attempt to fit it with simpler analytic forms, which can be fed into our iterative methods above as better initial guess (improvement to y_1 and $z_1$above) for superior numerical convergence. Let $x = 1 - e^{-c}$, x is in the range of 0 and 1. $y(x)$ is very smooth except when x is near 1 (That is, when c is large). Thus we can use polynomial approximation $y(x) = 1 + \sum_{k=1}^{n} (-1)^k B_k x^k$. When $n = 9$, the error of the fit is smaller than10^{-4} when $x < 0.934$ (that is, $c < e$). The result is listed in Table 4.

Table 4: The parameters B_k in the polynomial approximation of $y(x)$ in the symmetric lognormal cascade distribution

$B_1 = 1.0011$
$B_2 = 1.0515$
$B_3 = 2.2746$
$B_4 = 7.7488$
$B_5 = 24.3254$
$B_6 = 50.0072$
$B_7 = 61.3781$
$B_8 = 40.7858$
$B_9 = 11.3301$

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